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THE CANONICAL AND ALTERNATE DUALS OF A WAVELET FRAME

MARCIN BOWNIK AND JAKOB LEMVIG

Abstract. We show that there exists a frame wavelet ψ with fast decay in the time domain and compact support in the frequency domain generating a wavelet system whose canonical dual frame cannot be generated by an arbitrary number of generators. On the other hand, there exists infinitely many alternate duals of ψ generated by a single function. Our argument closes a gap in the original proof of this fact by Daubechies and Han [Appl. Comp. Harmonic Anal. **12** (2002), no. 3, 269–285].

1. Introduction

This paper explores the relationship between canonical and alternate dual frames of a wavelet frame. One of the first results in this direction is due to Daubechies [9] and Chui and Shi [7] who proved that the canonical dual of a wavelet frame need not have a wavelet structure. Since their example involved a non-biorthogonal Riesz wavelet, it has no alternate dual wavelet frames as well.

In general, if the canonical dual of a frame wavelet has a wavelet structure, then it is quite likely that this frame wavelet has some other wavelet duals. However, the existence of dual wavelet frames does not necessarily imply that the canonical dual must have a wavelet structure. This claim was asserted by Daubechies and Han [10].

Theorem 1. *There exists a frame wavelet $\psi \in L^2(\mathbb{R})$ such that:*

(i) $\hat{\psi}$ is C^∞ and compactly supported,
(ii) its canonical dual frame is not a wavelet system generated by a single function,
(iii) there are infinitely many $\tilde{\psi}$ such that ψ and $\tilde{\psi}$ form a pair of dual frame wavelets.

Unfortunately, the original argument in [10] uses an incorrect formula for the frame operator of a wavelet system owing to a simple change of sign mistake. This invalidates the original proof to the extent that an easy remedy appears to be doubtful. More details about the nature of this problem can be found in Section 3.

Therefore, there is a need to provide an alternative proof of Theorem 1. We will use a completely different approach motivated by [5]. Instead of trying to work directly with the frame operator as in [10], we will use a less direct approach using the following result of Weber and the first author [5].

Theorem 2. *Suppose that the canonical dual of a wavelet frame $\{\psi_{j,k}(x) := 2^{j/2}\psi(2^j x - k) : j,k \in \mathbb{Z}\}$ has a wavelet structure, i.e., it is of the form $\{\phi_{j,k} : j,k \in \mathbb{Z}\}$ for some frame wavelet ϕ. Then, the space of negative dilates*

$$V(\psi) := \overline{\operatorname{span}}\{\psi_{j,k} : j < 0,\ k \in \mathbb{Z}\} \tag{1}$$

is shift-invariant (SI).

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The paper is organized as follows. In Section 2 we recall some basic facts about the period of a wavelet frame. In particular, we explore the relationship between the period and the number of generators of the canonical dual of a wavelet frame. In Section 3 we give an explicit construction of a frame wavelet ψ as in Theorem 1. We prove that its corresponding space of negative dilates $V(\psi)$ lacks shift-invariance. Consequently, by Theorem 2 we conclude that the canonical dual of the wavelet frame $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is not a wavelet system generated by a single function. In fact, we prove that our example can be adjusted in such a way that the canonical dual can not be generated by arbitrarily many generators, see Theorem 3.

Finally, we review basic definitions. A frame for a separable Hilbert space $\mathcal{H}$ is a collection of vectors $\{f_j\}_{j\in\mathbb{J}}$, indexed by a countable set, such that there are constants $0 < C_1 \leq C_2 < \infty$ satisfying

$$C_1 \|f\|^2 \leq \sum_{j\in\mathbb{J}} |\langle f, f_j\rangle|^2 \leq C_2 \|f\|^2 \qquad \text{for all } f \in \mathcal{H}.$$

If the upper bound holds in the above inequality, then $\{f_j\}$ is said to be a Bessel sequence with Bessel constant C_2. The frame operator of $\{f_j\}$ is given by

$$S\colon \mathcal{H} \to \mathcal{H}, \qquad Sf = \sum_{j\in\mathbb{J}} \langle f, f_j\rangle f_j.$$

This operator is bounded, invertible, and positive. A frame $\{f_j\}$ is said to be *tight* if we can choose $C_1 = C_2$; this is equivalent to $S = C_1 I$, where I is the identity operator.

Two Bessel sequences $\{f_j\}$ and $\{g_j\}$ are said to be *dual frames* if

$$f = \sum_{j\in\mathbb{J}} \langle f, g_j\rangle f_j \qquad \text{for all } f \in \mathcal{H}.$$

It can be shown that two such Bessel sequences indeed are frames, and we shall say that the frame $\{g_j\}$ is *dual* to $\{f_j\}$, and vice versa. At least one dual always exists, it is given by $\{S^{-1} f_j\}$ and called the canonical dual. Redundant frames have several duals; a dual which is not the canonical dual is called an *alternate* dual.

Let $f \in L^2(\mathbb{R})$. Define dilation operator $D_a f(x) = |a|^{1/2} f(ax)$, translation operator $T_b f(x) = f(x-b)$, and modulation operator $E_c f(x) = e^{2\pi i c x} f(x)$, where $|a| > 1$, $b, c \in \mathbb{R}$. In the dyadic case we let $D := D_2$. The wavelet system generated by $\Psi = \{\psi_1, \dots, \psi_L\}$, is defined as $\{\psi_{j,k}\}_{j,k\in\mathbb{Z},\psi\in\Psi}$, where $\psi_{j,k} = D_a^j T_k \psi$. We say that Ψ and Φ is a pair of dual frame wavelets if their wavelet systems are dual frames. As stated above the canonical dual of a wavelet frame generated by Ψ might not be a wavelet system generated by $|\Psi|$ functions. In this case, we say that the canonical dual of Ψ does not have the wavelet structure.

Given a frame wavelet Ψ, the subspaces $W_j(\Psi)$ are defined by

$$W_j(\Psi) = \overline{\operatorname{span}} \{\psi_{j,k} : k \in \mathbb{Z}, \psi \in \Psi\}, \qquad j \in \mathbb{Z}. \tag{2}$$

By this definition we can write the space of negative dilates, introduced in Theorem 2, as

$$V(\Psi) = \overline{\operatorname{span}} \bigcup_{j<0} W_j(\Psi).$$

If we have only one generator, that is $L = 1$, we shall write $V(\psi)$ instead of $V(\Psi)$. Suppose that $W \subset L^2(\mathbb{R})$ is a closed subspace. We say W is $M\mathbb{Z}$-SI, $M\mathbb{Z}$ shift invariant, or shift invariant under $M\mathbb{Z}$, $M \in \mathbb{R}$, if $T_{Mz} W \subset W$ for all $z \in \mathbb{Z}$. In the case $M = 1$, we shall say that W is shift invariant, or SI.

For $f \in L^1(\mathbb{R})$, the Fourier transform is defined by $\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i \xi x}dx$ with the usual extension to $L^2(\mathbb{R})$. Given a measurable subset $K \subset \mathbb{R}$, we define the space $\check{L}^2(K)$, which is invariant under all translations, by

$$\check{L}^2(K) = \{f \in L^2(\mathbb{R}) : \operatorname{supp} \hat{f} \subset K\}.$$

2. The period of a frame wavelet

Daubechies and Han [10] have introduced the notion of the period of a dyadic wavelet frame in $L^2(\mathbb{R})$. Weber and the first author [5] extended it to a non-dyadic situation as below.

Definition 1. Suppose that $\Psi = \{\psi_1, \ldots, \psi_L\} \subset L^2(\mathbb{R})$ is a frame wavelet associated with an integer dilation factor a, $|a| \geq 2$. The *period* of Ψ is the smallest integer $p \geq 1$ such that for all $f \in \overline{\operatorname{span}}\,\{T_k\psi : k \in \mathbb{Z}, \psi \in \Psi\}$,

$$T_{pk}S^{-1}f = S^{-1}T_{pk}f \qquad \text{for all } k \in \mathbb{Z},$$

where S is the frame operator of the wavelet frame generated by Ψ. If there is no such p, we say that the period of Ψ is ∞.

We remark that our convention differs from the definitions in [5, 10], where the period is said to be 0 (and not ∞) if no such p exists. The examples of non-biorthogonal Riesz wavelets by Daubechies [9] and Chui and Shi [7] mentioned in the introduction have period ∞; while any tight frame wavelet has period 1.

Following [15], the local commutant of a system of operators $\mathcal{A}$ at the point $f \in L^2(\mathbb{R})$ is defined as

$$\mathcal{C}_f(\mathcal{A}) := \left\{B \in \mathcal{B}(L^2(\mathbb{R})) : BAf = ABf \quad \forall\, A \in \mathcal{A}\right\}.$$

The wavelet system of unitaries is denoted by $\mathcal{U} := \{D_a^jT_k : j \in \mathbb{Z}, k \in \mathbb{Z}\}$. The canonical dual of a wavelet frame $\mathcal{U}(\Psi) = \{D_a^jT_k\psi\}_{j,k\in\mathbb{Z},\psi\in\Psi}$ is given as

$$\begin{aligned}\left\{S^{-1}D_a^jT_k\psi_i : j,k \in \mathbb{Z}, i = 1,\ldots,L\right\} &= \left\{D_a^jS^{-1}T_k\psi_i : j,k \in \mathbb{Z}, i = 1,\ldots,L\right\} \\ &= \left\{D_a^j\eta^{k,i} : j,k \in \mathbb{Z}, i = 1,\ldots,L\right\},\end{aligned}$$

where S is the frame operator of $\mathcal{U}(\Psi)$, and $\{\eta^{k,i}\}$ is a family of functions, not necessarily with translation structure, indexed by $\{1,\ldots,L\} \times \mathbb{Z}$. The canonical dual takes the form of a wavelet system generated by $|\Psi| = L$ functions, *i.e.,*

$$\begin{aligned}\left\{S^{-1}D_a^jT_k\psi_i : j,k \in \mathbb{Z}, i = 1,\ldots,L\right\} &= \left\{D_a^jT_k(S^{-1}\psi_i) : j,k \in \mathbb{Z}, i = 1,\ldots,L\right\} \\ &= \left\{D_a^jT_k\phi_i : j,k \in \mathbb{Z}, i = 1,\ldots,L\right\},\end{aligned}$$

precisely when $T_kS^{-1}\psi = S^{-1}T_k\psi$ for all $\psi \in \Psi$ and $k \in \mathbb{Z}$; that is, precisely when $S^{-1} \in \mathcal{C}_\psi(\{T_k : k \in \mathbb{Z}\})$ for all $\psi \in \Psi$. Equivalently, the canonical dual of $\mathcal{U}(\Psi)$ has the wavelet structure generated by $|\Psi|$ functions if, and only if the period of Ψ is one, *c.f.* Proposition 2 below.

The following results from [5] will be used in the proof of Theorem 1. We restate them here since they were incorrectly stated in [5]. We note that these results can be thought as refinements of Theorem 2.

Proposition 1. *Let $M \in \mathbb{N}$. If Ψ is a frame wavelet and the period of Ψ divides M, then $V(\Psi)$ is shift invariant by the lattice $M\mathbb{Z}$. In addition, if Ψ is a Riesz wavelet, then the period of Ψ divides M if and only if $V(\Psi)$ is shift invariant by the lattice $M\mathbb{Z}$.*

Corollary 1. *If Ψ is a frame wavelet and the period of Ψ divides $|a|^J$ for some $J \geq 0$, then $D_a^J(V(\Psi))$ is shift invariant.*

If the period $P(\Psi)$ of a frame wavelet Ψ is finite, then the canonical dual frame is a wavelet system generated by $P(\Psi)\cdot|\Psi|$ functions, and this is the least number of generators. In this case the wavelet structure of the canonical dual frame is altered since it is based on the translation lattice $P(\Psi)\cdot\mathbb{Z}$ which is sparser than the original lattice $\mathbb{Z}$. Moreover, for any nonnegative integer M, the period of Ψ divides M if, and only if the canonical dual is a wavelet system generated by $M\cdot|\Psi|$ functions, see the proposition below. The "only if" direction is implicitly contained in the proof of [5, Proposition 2]. For the sake of completeness we prove both directions here.

Proposition 2. *Suppose that $\Psi = \{\psi_1, \dots, \psi_L\} \subset L^2(\mathbb{R})$ is a frame wavelet. For any nonnegative integer $M \in \mathbb{N}$, the following statements are equivalent:*

(i) $P(\Psi) \mid M$, i.e., the period of Ψ, denoted $P(\Psi)$, divides M.

(ii) There exist ML functions $\Phi = \{\phi_1, \dots, \phi_{ML}\}$ such that $\{D_a^j T_{Mk}\phi\}_{j,k\in\mathbb{Z},\phi\in\Phi}$ is the canonical dual of $\{D_a^j T_k\psi\}_{j,k\in\mathbb{Z},\psi\in\Psi} = \{D_a^j T_{Mk}\psi\}_{j,k\in\mathbb{Z},\psi\in\Psi_M}$, where

$$\Psi_M := \{T_m\psi : m = 0, \dots, M-1, \psi \in \Psi\}.$$

Proof. We note that the frame operator of $\{D_a^j T_k\psi\}_{j,k\in\mathbb{Z},\psi\in\Psi}$ equals the frame operator of $\{D_a^j T_{Mk}\psi\}_{j,k\in\mathbb{Z},\psi\in\Psi_M}$ since the two frames are setwise identical; we denote this operator by S.

We first prove (i) $\Rightarrow$ (ii). By assumption the period of Ψ is finite, hence the definition of the period yields the following equation.

$$T_{P(\Psi)k}S^{-1}f = S^{-1}T_{P(\Psi)k}f \quad \text{for all } k \in \mathbb{Z} \text{ and } f \in W_0(\Psi). \tag{3}$$

Since the period of Ψ divides M, we in particular have $P(\Psi)\mathbb{Z} \supset M\mathbb{Z}$, and the above equation gives us

$$T_{Mk}S^{-1}f = S^{-1}T_{Mk}f \quad \text{for all } k \in \mathbb{Z} \text{ and } f \in W_0(\Psi).$$

Consequently, for each $\psi \in \Psi$,

$$S^{-1}T_k\psi = S^{-1}T_{Ml}(T_m\psi) = T_{Ml}S^{-1}(T_m\psi),$$

where $k \in \mathbb{Z}$ is written as $k = Ml + m$ for $l \in \mathbb{Z}$ and $m \in \{0, 1, \dots, M-1\}$. The last equality in the above equation shows that $S^{-1} \in C_f(\{T_{Mk} : k \in \mathbb{Z}\})$ for every $f \in \Psi_M$, so we arrive at (ii) by taking $\Phi = S^{-1}\Psi_M = \{S^{-1}T_m\psi : m = 0, \dots, M-1, \psi \in \Psi\}$.

To prove the other direction, (ii) $\Rightarrow$ (i), we assume that the canonical dual of the system $\{D_a^j T_{Mk}\psi\}_{j,k\in\mathbb{Z},\psi\in\Psi_M}$ is generated by ML functions $\Phi = \{\phi_1, \dots, \phi_{ML}\}$. Since $|\Psi_M| = ML$, it follows that $S^{-1} \in C_\psi(\{T_{Mk} : k \in \mathbb{Z}\})$ for all $\psi \in \Psi_M$, i.e.,

$$S^{-1}T_{Mk}(T_m\psi) = T_{Mk}S^{-1}(T_m\psi) \quad \text{for all } k \in \mathbb{Z},\, m \in \{0, \dots, M-1\}, \psi \in \Psi. \tag{4}$$

In this equation we replace $k \in \mathbb{Z}$ by $k+l$ with $l \in \mathbb{Z}$, whereby we obtain $S^{-1}T_{Mk}(T_{Ml+m}\psi) = T_{Mk}S^{-1}(T_{Ml+m}\psi)$ for all $k, l \in \mathbb{Z}$, $m \in \{0, \dots, M-1\}$, and $\psi \in \Psi$. Now since

$$W_0(\Psi) = \overline{\operatorname{span}}\{T_{Ml+m}\psi : l \in \mathbb{Z},\ m \in \{0, \dots, M-1\},\ \psi \in \Psi\},$$

we see that

$$S^{-1}T_{Mk}f = T_{Mk}S^{-1}f \qquad \text{for all } k \in \mathbb{Z},\ f \in W_0(\Psi), \tag{5}$$

and conclude that the period of Ψ is at most M.

To complete the proof we need to show that the period of Ψ is a divisor of M. Assume on the contrary that the period of Ψ is not a divisor of M. Then there are $q, r \in \mathbb{N} \cup \{0\}$ such that $M = qP(\Psi) + r$ and $0 < r < P(\Psi)$. We know that the period of Ψ is finite, so equation (3) is satisfied, and by from (3) and (5) we have

$$S^{-1}T_{P(\Psi)k_1+Mk_2}f = T_{P(\Psi)k_1+Mk_2}S^{-1}f \quad \text{for } k_1, k_2 \in \mathbb{Z},\ f \in W_0(\Psi).$$

Taking $k_1 = -qk$ and $k_2 = k$ for each $k \in \mathbb{Z}$ gives us $rk = P(\Psi)k_1 + Mk_2$. Therefore,

$$S^{-1}T_{rk}f = T_{rk}S^{-1}f \qquad \text{for all } k \in \mathbb{Z},\ f \in W_0(\Psi),$$

which contradicts the minimality of $P(\Psi)$ since $0 < r < P(\Psi)$. □

Remark 1. In the dyadic case and when M is a power of two, Proposition 2 reduces to [10, Proposition 2.1]. Indeed, if $M = 2^J$ for some $J \in \mathbb{N}$, then any dyadic wavelet system of the form $\{D^jT_{Mk}\phi\}_{j,k\in\mathbb{Z},\phi\in\Phi}$ with translation with respect to the lattice $M\mathbb{Z}$, can be written as a wavelet system $\{D^jT_k\phi\}_{j,k\in\mathbb{Z},\phi\in\Phi'}$ using the standard translation lattice $\mathbb{Z}$ and the same number of generators $|\Phi| = |\Phi'|$, see [10]. Corollary 7 in [5] states that the period of a dyadic Riesz wavelet is either a power of two or infinite. Hence, whenever a Riesz wavelet has finite period the canonical dual takes the form $\{D^jT_k\phi\}_{j,k\in\mathbb{Z},\phi\in\Phi'}$ for some family of functions Φ', where we note that the translation is with respect to the lattice $\mathbb{Z}$.

3. Canonical dual frames without wavelet structure

In this section we will prove Theorem 1 by giving an example of a wavelet frame in $L^2(\mathbb{R})$ whose canonical dual does not have wavelet structure. To be precise, we will construct a family of examples, indexed by $J \in \mathbb{N}$, such that the canonical dual cannot be generated by fewer than 2^J functions. In each of these examples the wavelet itself is nice in the sense that it has compact support in the Fourier domain and fast decay in the time domain, and it has nice alternate dual frame wavelets.

Our construction is motivated by the proof of [5, Theorem 2(ii)], where Weber and the first author give an example of a frame wavelet ψ with compact support in the Fourier domain whose canonical dual cannot be generated by one function. The Fourier transform of ψ is not continuous yielding poor decay in the time domain. Furthermore, the space of negative dilates $V(\psi)$ is not $\mathbb{Z}$-SI (this is necessary in order to utilize Theorem 2), but it is in fact $2\mathbb{Z}$-SI, hence the canonical dual must be generated by at least two functions, *c.f.* Proposition 1. We modify this example so that $\hat{\psi}$ becomes C^∞ and so that the space of negative dilates becomes non $p\mathbb{Z}$-SI for $p < 2^J$ and $p \in \mathbb{N}$ for a chosen $J \in \mathbb{N}$. Hence, we have the following generalization of Theorem 1.

Theorem 3. *For all $J \in \mathbb{N}$, there exists a frame wavelet $\psi \in L^2(\mathbb{R})$ such that:*

(i) $\hat{\psi}$ is C^∞ and compactly supported,
(ii) its canonical dual frame is not a wavelet system generated by fewer than 2^J function,
(iii) there are infinitely many $\tilde{\psi}$ such that ψ and $\tilde{\psi}$ form a pair of dual wavelet frames.

Before providing the proof of Theorem 3, we will analyze the original proof of Theorem 1 by Daubechies and Han [10]. The key role in the argument of [10] is played by an explicit formula for the frame operator of a wavelet system.

Proposition 3. *Suppose that $\Psi = \{\psi_1, \dots, \psi_L\} \subset L^2(\mathbb{R})$ generates a wavelet system which is a Bessel sequence. Let*

$$\mathcal{D} = \{f \in L^2(\mathbb{R}) : \hat{f} \in L^\infty(\mathbb{R}) \text{ and } \operatorname{supp} \hat{f} \subset [-R, -1/R] \cup [1/R, R] \qquad \text{for some } R > 1\}.$$

Then its frame operator S is given by

$$\widehat{Sf}(\xi) = \hat{f}(\xi)\sum_{l=1}^{L}\sum_{j\in\mathbb{Z}}|\hat{\psi}_l(2^j\xi)|^2 + \sum_{p\in\mathbb{Z}}\sum_{q\in 2\mathbb{Z}+1}\hat{f}(\xi+2^{-p}q)t_q(2^p\xi) \qquad \text{for a.e. } \xi\in\mathbb{R}, \tag{6}$$

and for all $f \in \mathcal{D}$, where

$$t_q(\xi) = \sum_{l=1}^{L}\sum_{j=0}^{\infty}\hat{\psi}_l(2^j\xi)\overline{\hat{\psi}_l(2^j(\xi+q))} \qquad \text{for } q\in\mathbb{Z}.$$

Proposition 3 is implicitly contained in the book of Hernández and Weiss [16, Proposition 7.1.19]. This result can be extended to higher dimensions and more general dilations, see [4, 13, 14, 18].

Initially, the problem with the argument of Daubechies and Han appears to be very minor since the formula (2.6) of [10] lacks a negative sign which is present in $\hat{f}(\xi+2^{-p}q)$ of (6). This mistake can be traced back to the proof of Lemma 2.3 in [14]. However, this change of sign has profound effects for the rest of this paper. First, it affects Lemma 3.1 in [10] by wiping out the negative signs in $2^{-j}K_1$ and $2^{-j}K_2$ of formula (3.1). Consequently, it invalidates the proof of [10, Theorem 3.3]. To see this, consider the example borrowed from the paper of Weber and the first author [5].

Example 1. Let $\psi_b \in L^2(\mathbb{R})$ be given by

$$\hat{\psi}_b = \chi_{[-1,-b]\cup[b,1]}.$$

In [5] it is shown that ψ_b is a biorthogonal Riesz wavelet whenever $1/3 \le b \le 1/2$. In fact, one can explicitly exhibit its dual biorthogonal wavelet ϕ_b as

$$\hat{\phi}_b = \chi_{[-1,-1/2]\cup[1/2,1]} - \chi_{[-2+2b,-1]\cup[1,2-2b]}.$$

We note that this fact is far from being obvious, since one can also show that ψ_b is **not** a frame wavelet when $1/6 < b < 1/3$, see [5, Example 2]. While ψ_b is of a slightly different form than the function considered in [10, Theorem 3.3], one could arrive at the conclusion that ψ_b is not a biorthogonal wavelet when $b = 1/3$ by following the same argument as in [10]. This stands in a direct contradiction with the above mentioned fact from [5]. In fact, this is how the change of sign mistake in [10] was uncovered by the first author.

In order to prove Theorem 3 we need to show two lemmas.

Lemma 1. *For every $N \ge 4$ and $0 < \delta < 2^{-N}$, there exists a frame wavelet ψ such that $\hat{\psi} \in C_0^\infty(\mathbb{R})$ and*

$$\begin{aligned}\hat{\psi}(\xi) \neq 0 \iff \xi &\in (-1/2,-1/4)\cup(1/2,3/4) \\ &\cup\left(-2^{-N+1}-\delta,-2^{-N}+\delta\right)\cup\left(2^{-N}-\delta,2^{-N+1}+\delta\right)\end{aligned} \tag{7}$$

$$\hat{\psi}(\xi) = \hat{\psi}(\xi-1) \neq 0 \qquad \text{for } \xi\in(1/2,3/4)\,. \tag{8}$$

Proof. Let $\psi^0 \in L^2(\mathbb{R})$ be a frame wavelet such that $\hat{\psi}^0 \in C_0^\infty(\mathbb{R})$ and

$$\hat{\psi}^0(\xi)\neq 0 \iff \xi\in\left(-2^{-N+1}-\delta,-2^{-N}+\delta\right)\cup\left(2^{-N}-\delta,2^{-N+1}+\delta\right),$$

where $N \ge 4$ and $0 < \delta < 2^{-N}$ as in the assumption. Let $\psi^1 \in L^2(\mathbb{R})$ be such that $\hat{\psi}^1 \in C_0^\infty(\mathbb{R})$ has support in $[-1/2,-1,4]\cup[1/2,3/4]$ and

$$\hat{\psi}^1(\xi) = \hat{\psi}^1(\xi-1)\neq 0 \qquad \text{whenever } \xi\in(1/2,3/4)\,. \tag{9}$$

For any such $\psi^1 \in L^2(\mathbb{R})$ the sequence $\{D^j T_k \psi^1\}$ generates a Bessel sequence by [17, Theorem 13.0.1] or by the proof of [8, Lemma 3.4].

Define $\psi \in L^2(\mathbb{R})$ by $\psi = \psi^0 + \varepsilon\psi^1$, where $\varepsilon\psi^1$ acts as a perturbation on the wavelet frame generated by ψ^0 and ensures that ψ satisfies (8), see also Figure 1. Denote the frame bounds of $\{D^j T_k \psi^0\}$ by C_1 and C_2, and the Bessel bound of $\{D^j T_k \psi^1\}$ by C_0. The function $\varepsilon\psi^1$ generates a Bessel sequence with bound $\varepsilon^2 C_0$. Hence, for sufficiently small $\varepsilon > 0$, we have $\varepsilon^2 C_0 < C_1$, and by a perturbation result [6, Corollary 2.7] or [12, Theorem 3], we conclude that ψ generates a wavelet frame. By our construction $\hat{\psi}$ is in $C_0^\infty(\mathbb{R})$ and satisfies (7) and (8).

Finally, let us illustrate how one can construct two such functions ψ^0 and ψ^1. For $N \geq 4$ and $0 < \delta < 2^{-N}$, define the function η by

$$\hat{\eta} = h_\delta * \chi_{[-2^{-N+1}, -2^{-N}] \cup [2^{-N}, 2^{-N+1}]}, \tag{10}$$

where $h_\delta(x) = \delta^{-1} h(x/\delta)$ with $h \in C_0^\infty(\mathbb{R})$, $h \geq 0$, $\int_{\mathbb{R}} h(x)dx = 1$, and $\operatorname{supp} h \subset [-1, 1]$. This yields $\hat{\eta} \in C^\infty$ with

$$\hat{\eta}(\xi) \neq 0 \iff \xi \in \left(-2^{-N+1} - \delta, -2^{-N} + \delta\right) \cup \left(2^{-N} - \delta, 2^{-N+1} + \delta\right).$$

By $\|\hat{\eta}\|_{L^\infty} \leq 1$ and the above, there exist constants $C_1, C_2 > 0$, such that

$$0 < C_1 \leq \sum_{j\in\mathbb{Z}} \left|\hat{\eta}(2^j\xi)\right|^2 \leq C_2 < 2 \quad \text{for all } \xi \in \mathbb{R} \setminus \{0\}.$$

Moreover, for $q \in 2\mathbb{Z} + 1$,

$$t_q(\xi) := \sum_{j=0}^{\infty} \hat{\eta}(2^j\xi)\overline{\hat{\eta}(2^j(\xi + q))} = 0 \quad \text{for all } \xi \in \mathbb{R},$$

since $\hat{\eta}(2^j\cdot)$ and $\hat{\eta}(2^j(\cdot + q))$ have disjoint support for all $j \geq 0$. We define ψ^0 as a normalization of η by

$$\hat{\psi}^0(\xi) = \frac{\hat{\eta}(\xi)}{\sqrt{\sum_{j\in\mathbb{Z}} |\hat{\eta}(2^j\xi)|^2}} \quad \text{for } \xi \in \mathbb{R} \setminus \{0\}, \tag{11}$$

and $\hat{\psi}^0(0) = 0$. Consequently, we have $\sum_{j\in\mathbb{Z}} |\hat{\psi}^0(2^j\xi)|^2 = 1$ and $t_q(\xi) = 0$ for $\xi \in \mathbb{R}$ and $q \in 2\mathbb{Z} + 1$. By [16, Theorem 7.1.6], ψ^0 generates a tight wavelet frame with frame bound 1, and it has the desired properties. For the proof of the lemma the last normalization step could be omitted since η itself generates a (non-tight) frame. However, it is included since we later want to use the fact that the ψ^0 can be chosen to be a tight frame wavelet with frame bound 1.

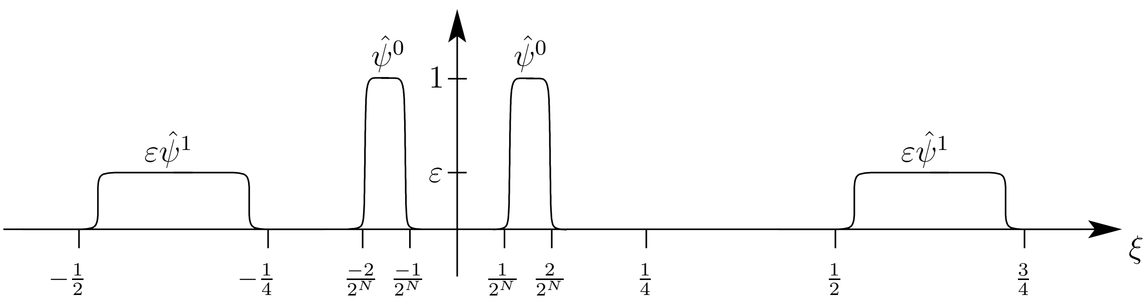

FIGURE 1. Sketch of the graph of $\hat{\psi} = \hat{\psi}^0 + \varepsilon\hat{\psi}^1$.

The construction of the perturbation term ψ^1 is straightforward. Let $\theta_\lambda := h_\lambda * \chi_{[1/2+\lambda,3/4-\lambda]}$ for some $0 < \lambda < 1/8$, where h_λ is defined as above. Define ψ^1 by $\hat{\psi}^1 = \theta_\lambda + T_{-1}\theta_\lambda$. This makes $\hat{\psi}^1$ a C^∞ function with compact support in $[-1/2,-1,4]\cup[1/2,3/4]$, satisfying equation (9). This completes the proof of Lemma 1. ☐

Lemma 2. *Suppose that a function $\psi \in L^2(\mathbb{R})$ satisfies* (7) *and* (8) *for some $N \geq 4$ and $0 < \delta < 2^{-N}$. Then, the space of negative dilates $V(\psi)$ is* **not** *$p\mathbb{Z}$-SI for any $p < 2^{N-3}$, $p \in \mathbb{N}$.*

Proof. To prove this claim we will look at the subspaces $W_j(\psi)$ for $j \leq 0$, defined by

$$W_j(\psi) = \overline{\operatorname{span}}\Big\{D^j T_k \psi : k \in \mathbb{Z}\Big\}, \qquad j \in \mathbb{Z}.$$

First, consider a principal shift-invariant (PSI) subspace $W_0(\psi) = \overline{\operatorname{span}}\{T_k\psi\}_{k\in\mathbb{Z}}$. By a result in [11], this subspace can be described as

$$W_0(\psi) = \{f \in L^2(\mathbb{R}) : \hat{f} = \hat{\psi} m \quad \text{for some measurable, 1-periodic } m\}.$$

Hence, by (7) and (8) we have

$$\begin{aligned} W_0(\psi) = \Big\{ f \in L^2(\mathbb{R}) : \operatorname{supp}\hat{f} \subset [-1/2,-1/4]\cup[1/2,3/4]\cup K \\ \hat{f}(\xi-1) = \hat{f}(\xi) \quad a.e.\ \xi \in [1/2,3/4] \Big\}, \end{aligned} \tag{12}$$

$$\text{where } K = \Big[-2^{-N+1}-\delta, -2^{-N}+\delta\Big] \cup \Big[2^{-N}-\delta, 2^{-N+1}+\delta\Big].$$

Applying the scaling relation $W_j(\psi) = D^j W_0(\psi)$ to (12) yields

$$\begin{aligned} W_j(\psi) = \Big\{ f \in L^2(\mathbb{R}) : \operatorname{supp}\hat{f} \subset \Big[-2^{j-1},-2^{j-2}\Big] \cup \Big[2^{j-1}, 3/2\cdot 2^{j-1}\Big] \cup 2^j K, \\ \hat{f}(\xi-2^j) = \hat{f}(\xi) \quad a.e.\ \xi \in \Big[2^{j-1}, 3/2\cdot 2^{j-1}\Big] \Big\}. \end{aligned} \tag{13}$$

Therefore, each space $W_j(\psi)$, $j \in \mathbb{Z}$, can be decomposed as the orthogonal sum

$$W_j(\psi) = W_j^0 \oplus W_j^1, \qquad \text{where} \tag{14}$$

$$W_j^0 = \check{L}^2(2^j K), \tag{15}$$

$$\begin{aligned} W_j^1 = \Big\{ f \in L^2(\mathbb{R}) : \operatorname{supp}\hat{f} \subset \Big[-2^{j-1},-2^{j-2}\Big] \cup \Big[2^{j-1}, 3/2\cdot 2^{j-1}\Big], \\ \hat{f}(\xi-2^j) = \hat{f}(\xi) \quad a.e.\ \xi \in \Big[2^{j-1}, 3/2\cdot 2^{j-1}\Big] \Big\}. \end{aligned} \tag{16}$$

Using (14), it is possible to describe the space of negative dilates

$$V(\psi) = \overline{\operatorname{span}}\Big(\bigcup_{j<0} W_j(\psi)\Big)$$

in the Fourier domain. However, such a description would be quite complicated owing to interactions of the spaces W_j^0 and W_j^1 at various scales $j < 0$.

Instead, we consider another space

$$\widetilde{V}(\psi) = V(\psi) \cap \check{L}^2((-\infty, -2^{-N+1}] \cup [2^{-N+2}, \infty)).$$

By (15) and $K \subset (-2^{-N+2}, 2^{-N+2})$, we have

$$W_j^0 \subset \check{L}^2([-2^{-N+1}, 2^{-N+2}]) \qquad \text{for } j < 0.$$

Likewise, by (16) we have

$$W_j^1 \subset \begin{cases} \check{L}^2([-2^{-N+1}, 2^{-N+2}]) & \text{for } j \le -N+2, \\ \check{L}^2((-\infty, -2^{-N+1}] \cup [2^{-N+2}, \infty)) & \text{for } j \ge -N+3. \end{cases}$$

Combining the last four equations with (14) yields

$$\widetilde{V}(\psi) = \overline{\operatorname{span}}\Big(\bigcup_{j<0} W_j(\psi) \cap \check{L}^2((-\infty, -2^{-N+1}] \cup [2^{-N+2}, \infty))\Big) = \overline{\operatorname{span}}\Big(\bigcup_{j=-N+3}^{-1} W_j^1\Big),$$

and further, by the orthogonality of the subspaces $W^1_{-N+3}, \dots, W^1_{-1}$,

$$\widetilde{V}(\psi) = \bigoplus_{j=-N+3}^{-1} W_j^1.$$

Consequently, by (16),

$$\begin{aligned} \widetilde{V}(\psi) = \Big\{ f \in L^2(\mathbb{R}) : \operatorname{supp} \hat{f} \subset \bigcup_{j=-N+3}^{-1} 2^j([-1/2, -1/4] \cup [1/2, 3/4]), \\ \hat{f}(\xi - 2^{-1}) = \hat{f}(\xi) \quad a.e.\, \xi \in \left[2^{-2}, 3/2 \cdot 2^{-2}\right], \\ \hat{f}(\xi - 2^{-2}) = \hat{f}(\xi) \quad a.e.\, \xi \in \left[2^{-3}, 3/2 \cdot 2^{-3}\right], \\ \vdots \qquad\qquad\qquad \vdots \\ \hat{f}(\xi - 2^{-N+3}) = \hat{f}(\xi) \quad a.e.\, \xi \in \left[2^{-N+2}, 3/2 \cdot 2^{-N+2}\right] \Big\}. \end{aligned} \tag{17}$$

Assume, towards a contradiction, that $V(\psi)$ is $p\mathbb{Z}$-SI for some $p < 2^{N-3}$ with $p \in \mathbb{N}$. Then, $\widetilde{V}(\psi)$ is $p\mathbb{Z}$-SI as well. Define $f \in L^2(\mathbb{R})$ by

$$\hat{f} = \chi_{I_N \cup (I_N - 2^{-N+3})}, \qquad \text{where } I_N = [2^{-N+2}, 3/2 \cdot 2^{-N+2}].$$

Then $f \in \widetilde{V}(\psi)$, and by our hypothesis we have $T_{pk} f \in \widetilde{V}(\psi)$ for all $k \in \mathbb{Z}$. Equivalently, using $\mathcal{F} T_k = E_{-k} \mathcal{F}$, we have $E_{pk} \hat{f} \in \mathcal{F}(\widetilde{V}(\psi))$ for all $k \in \mathbb{Z}$. For $k = 1$, this implies that $E_p \hat{f}(\xi) = e^{2\pi i p \xi} \chi_{I_N \cup (I_N - 2^{-N+3})}(\xi) \in \mathcal{F}(\widetilde{V}(\psi))$. By (17),

$$e^{2\pi i p(\xi - 2^{-N+3})} = e^{2\pi i p \xi} \qquad \text{for a.e. } \xi \in I_N.$$

This can only be satisfied if $e^{-2\pi i p 2^{-N+3}} = 1$, which contradicts the hypothesis that $1 \le p < 2^{N-3}$. This completes the proof of Lemma 2. □

Remark 2. A more detailed analysis shows that $V(\psi)$ is $2^{N-2}\mathbb{Z}$-SI, and it is not shift invariant by any sublattice of $\mathbb{Z}$ strictly larger than $2^{N-2}\mathbb{Z}$. Since we do not need such precise assertion, we will skip its proof.

Finally, we are ready to complete the proof of Theorem 3.

Proof of Theorem 3. Take any $J \in \mathbb{N}$. Suppose that ψ is a frame wavelet as in Lemma 1 with $N = J + 3$. By Lemma 2 and Proposition 1, the period of ψ is at least 2^{N-3}. Hence, by Proposition 2, we need at least 2^J functions to generate the canonical dual of $\{D^j T_k \psi\}_{j,k \in \mathbb{Z}}$.

We have only left to show that the wavelet frame generated by ψ has infinitely many alternate duals that are generated by one function. For this purpose it is convenient to assume that $\psi = \psi^0 + \varepsilon \psi^1$ is of the same form as in the proof of Lemma 1, *i.e.,* ψ^0 generates a tight frame with frame bound 1. Hence, the functions ψ and ψ^0 satisfy the characteristic equations

$$\sum_{j\in\mathbb{Z}} \hat{\psi}(2^j\xi)\overline{\hat{\psi}^0(2^j\xi)} = 1, \qquad a.e.\, \xi \in \mathbb{R},$$
$$\sum_{j=0}^{\infty} \hat{\psi}(2^j\xi)\overline{\hat{\psi}^0(2^j(\xi+q))} = 0, \qquad a.e.\, \xi \in \mathbb{R} \text{ for odd } q \in \mathbb{Z},$$

since $\hat{\psi} = \hat{\psi}^0$ on $\operatorname{supp}\hat{\psi}^0$ and since $\hat{\psi}(2^j\cdot)\hat{\psi}^0(2^j(\cdot+q)) = 0$ for all $j \geq 0$ and all odd q. We conclude that $\{\psi^0_{j,k}\}$ is a dual frame of $\{\psi_{j,k}\}$. Since $\{\psi^0_{j,k}\}$ is generated by one function, it is apparent from the above that $\{\psi^0_{j,k}\}$ must be an alternate dual.

Any function $\phi \in L^2(\mathbb{R})$ defined by $\hat{\phi} = \hat{\psi}^0 + h$, where

$$h \in \mathbb{C}^\infty(\mathbb{R}), \qquad \operatorname{supp} h \subset [-1/4, 1/2], \qquad \operatorname{supp} h \cap \operatorname{supp}\hat{\psi}^0 = \emptyset, \qquad h(0) = 0,$$

generates a Bessel sequence by [17, Theorem 13.0.1]. Since ψ and ϕ satisfy the characteristic equations above, such a ϕ is an alternate dual frame wavelet of ψ. This example demonstrates that we have infinitely many alternate duals, and completes the proof of Theorem 3. □

We end by putting our example in a perspective with other known results.

Remark 3. Auscher [1] proved that every "regular" orthonormal wavelet $\psi \in L^2(\mathbb{R})$ is associated with an MRA. "Regular" means that $|\hat{\psi}|$ is continuous and $\hat{\psi}(\xi) = \mathcal{O}(|\xi|^{-1/2-\delta})$ as $|\xi| \to \infty$ for some $\delta > 0$, see [16, Corollary 7.3.16]. This fact does not hold for tight frame wavelets. In fact, Baggett et al. [2] constructed a non-MRA C^r tight frame wavelet with rapid decay for any $r \in \mathbb{N}$. Moreover, their tight frame wavelet is associated with a GMRA having the same dimension/multiplicity function as the Journé wavelet. Once we allow non-tight frame wavelets we might lose even the GMRA property. Indeed, the frame wavelet from Theorem 3 is an example of a non-GMRA C^∞ frame wavelet with rapid decay.

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